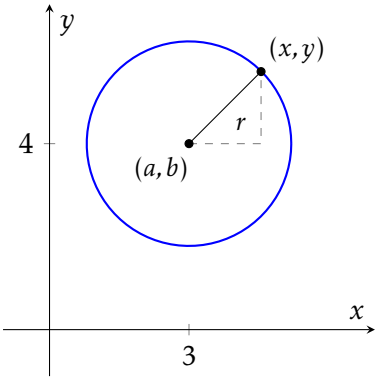
The Equation of a Circle

A circle is the set of points at a fixed distance r from a fixed centre. The equation comes straight from Pythagoras.

Theorem

Centre $(0,0)$, radius r : $\quad x^2 + y^2 = r^2$

Centre (a,b), radius r : $\quad (x-a)^2 + (y-b)^2 = r^2$



Example 1. Write down the centre and radius of $(x-3)^2 + (y+5)^2 = 49$.

2. Find the equation of the circle with centre $(2,-1)$ passing through $(4,-5)$.

3. Does the point $(7,1)$ lie inside, on, or outside the circle in part 2?

1. *Centre* $(3,-5)$, *radius 7. (Watch the signs.)*
2. $r^2 = (4-2)^2 + (-5+1)^2 = 20$: $\quad (x-2)^2 + (y+1)^2 = 20$
3. $(7-2)^2 + (1+1)^2 = 29 > 20$: *outside.*

Example

The points $A(-2,11)$ and $B(8,1)$ are the ends of a diameter of a circle. Find the equation of the circle.

Centre = midpoint $(3,6)$. $r^2 = (8-3)^2 + (1-6)^2 = 50$.
$(x-3)^2 + (y-6)^2 = 50$

Textbook Exercises: SPS Course 3.4, Exercise 0

The General Form

Expanding $(x-a)^2+(y-b)^2=r^2$ gives an equation of the form

$$x^2+y^2+2gx+2fy+c=0.$$

To find the centre and radius, complete the square in x and in y.

Example (Edexcel C2)
The circle C has equation $x^2+y^2-20x-24y+195=0$. Find the centre and radius of C.

$(x-10)^2-100+(y-12)^2-144+195=0$
$(x-10)^2+(y-12)^2=49$: *centre* $(10,12)$, *radius 7.*

Example
Show that the circle $(x-3)^2+(y-5)^2=4$ touches the circle $x^2+y^2-2y-8=0$.

Second circle: $x^2+(y-1)^2=9$: *centre* $(0,1)$, *radius 3. First: centre* $(3,5)$, *radius 2.*
Distance between centres $=\sqrt{3^2+4^2}=5=3+2=$ *sum of the radii, so the circles touch (externally).*

Example

Find the values of k for which $x^2 + y^2 + 6x - 4y + k = 0$ is the equation of a circle.

$(x+3)^2 + (y-2)^2 = 13 - k$: *need* $r^2 > 0$, *so* $k < 13$.

Textbook Exercises: SPS Course 3.4, Exercise 1

Tangents and Normals

Fact — The tangent to a circle at a point P is perpendicular to the radius at P. So:

$$\text{gradient of tangent at } P = -\frac{1}{\text{gradient of radius to } P}$$

The normal at P is the line through the centre.

Example (Edexcel C2)

A circle C with centre $(2,-1)$ passes through the point $A(4,-5)$.

1. Find an equation for C.
2. Find an equation of the tangent to C at A, in the form $ax+by+c=0$ for integers a, b, c.

1. $r^2 = 4+16 = 20$: $(x-2)^2+(y+1)^2 = 20$
2. *Radius gradient* $= \frac{-5+1}{4-2} = -2$, *so tangent gradient* $= \frac{1}{2}$.
   $y+5 = \frac{1}{2}(x-4) \implies x-2y-14 = 0$

Fact (Length of a tangent from an external point) — If the tangent from an external point N touches the circle (centre M, radius r) at P, then triangle MPN is right-angled at P:

$$NP^2 = MN^2 - r^2$$

Example (Edexcel C2)

The circle C has equation $x^2 + y^2 - 20x - 24y + 195 = 0$, with centre M. The tangent to C at a point P passes through $N(25, 32)$. Find the length NP.

From Lesson 2: $M(10, 12)$, $r = 7$.

$MN = \sqrt{15^2 + 20^2} = 25$. $NP = \sqrt{25^2 - 7^2} = \sqrt{576} = 24$.

Textbook Exercises: SPS Course 3.4, Exercise 2

Lines Meeting Circles

To find where a line meets a circle, substitute the line into the circle. The discriminant of the resulting quadratic counts the intersections:

$b^2 - 4ac > 0$ the line crosses the circle twice
$b^2 - 4ac = 0$ the line is a tangent
$b^2 - 4ac < 0$ the line misses the circle

Example
Find the points where the line $y = x + 1$ meets the circle $x^2 + y^2 = 25$.

$x^2 + (x+1)^2 = 25 \implies 2x^2 + 2x - 24 = 0 \implies x^2 + x - 12 = 0$
$(x+4)(x-3) = 0$: *points* $(-4,-3)$ *and* $(3,4)$.

Example
Find the values of k for which the line $y = 7x + k$ never intersects the circle $x^2 + y^2 = 2$.

$x^2 + (7x+k)^2 = 2 \implies 50x^2 + 14kx + k^2 - 2 = 0$
No intersection: $(14k)^2 - 200(k^2 - 2) < 0 \implies -4k^2 + 400 < 0 \implies k^2 > 100$
$k > 10$ *or* $k < -10$.

Exercise (Investigation)**.** Find the equation of the (unique) circle passing through $(0,0)$, $(6,0)$ and $(0,8)$. Then devise a general method for finding the circle through any three given points, and identify when no such circle exists.

Textbook Exercises: SPS Course 3.4, Problems 3.4.3